

Pricing distressed CDOs with Base Correlation and Stochastic Recovery

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Abstract

In February and March 2008 it was temporarily not possible to calibrate the standard Gaussian Base correlation model to the complete set of CDX.IG and ITRAXX.IG CDO tranche quotes. For instance, in CDX.IG it failed for the 15% - 30% senior tranche and hence the successive 30% - 100% tranche. The reason is that the Gaussian Base correlation model was not able to generate enough probability for high portfolio losses, while preserving the calibration to mezzanine and equity tranches. We introduce a Gaussian Base correlation model with correlated *stochastic recovery rates* to overcome this problem. Moreover, our model can be used to price super senior tranches (e.g. 60% - 100%), which have a fair spread of zero in a standard copula model with fixed recovery.

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1 Introduction

In February and March 2008 it was temporarily not possible to calibrate the Gaussian Base correlation model to CDX.IG and ITRAXX.IG CDO market quotes of senior tranches (.e.g 15% - 30% in CDX). The reason is that the Gaussian Base correlation was not able to generate enough probability for high portfolio losses while preserving the calibration to mezzanine and equity tranches. We introduce a Gaussian Base Correlation model with correlated *stochastic recovery rates* to overcome this calibration problem. We this extension the calibration range of the Base correlation model is significantly widen and in addition it will enable us to price super senior tranches with attachment points outside the range of fixed recoveries. Random recovery was already introduced by [1, Andersen, Sidenius (2004)]. They modelled the recovery rate as a function of the common market factor and additional idiosyncratic risk factors. In contradiction to this approach, in our model the recovery rate is driven by the default triggering factor which circumvents some technical issues. The discrete recovery rate distribution is user input and can arbitrarily be chosen.

The rest of the paper is organized as follows. In section 2 we shortly recapitulate the well-known Gaussian copula model and in section 3 we present our stochastic recovery extension. Section 4 contains the calibration results in distressed CDO markets and a derivation of the recovery correlation. Finally, we present our conclusion in section 5.

2 Standard Gaussian Copula Model

Let's assume the CDO index contains M obligors and the default probability of issuer m at time T_i is labeled with q_i^m , where T_i ($i = 1, \dots, I$) are the coupon payment dates (These probabilities are usually bootstrapped from the CDS quotes). The default triggering variable for X_i^m at time T_i for debtor m is modelled by

$$\begin{aligned} X_i^m &= \sqrt{\rho}Z_i^0 + \sqrt{1-\rho}Z_i^m, \\ X_i^m \leq c_i^m &\equiv \tau^m \leq T_i, \end{aligned}$$

where Z_i^0 is the systematic factor and Z_i^m ($m = 1, \dots, M$) are the idiosyncratic factors. These random variables are independent and identically Gaussian distributed. Hence, X_i^m is also standard Gaussian distributed (denoted by \mathcal{N}) and the correlation between two factors X_i^m and X_i^n is ρ . We call this correlation the *factor correlation*, not to shuffle with the real default correlation defined by $\text{COR}[1_{\{\tau_m \leq T_i\}}, 1_{\{\tau_n \leq T_i\}}]$, which is usually different. With

$$c_i^m := \mathcal{N}^{-1}(q_i^m)$$

we achieve

$$P(X_i^m \leq c_i^m) = q_i^m$$

and

$$\begin{aligned}
P(\tau_m \leq T_i | Z_i^0 = z) &\equiv P(X_i^m \leq c_i^m | Z_i^0 = z) \\
&= P(\sqrt{\rho}z + \sqrt{1-\rho}Z_i^m \leq c_i^m | Z_i^0 = z) \\
&= P\left(Z_i^m \leq \frac{c_i^m - \sqrt{\rho}z}{\sqrt{1-\rho}} \middle| Z_i^0 = z\right) \\
&= \mathcal{N}\left(\frac{c_i^m - \sqrt{\rho}z}{\sqrt{1-\rho}}\right).
\end{aligned}$$

If issuer m defaults, the portfolio notional is reduced by the nominal N^m times one minus the quoted recovery rate, namely $N^m(1 - \text{REC}^m)$. The cumulative loss L_i until T_i is thus given by

$$L_i = \sum_{m=1}^M N^m(1 - \text{REC}^m) \mathbf{1}_{\{\tau^m \leq T_i\}}.$$

Conditioned on $Z_i^0 = z$ the loss distribution is a convolution of independent Bernoulli variables. Methods to calculate respectively approximate this loss distribution can be found amongst others in [1, Andersen & Sidenius (2004)], [2, Hull & White (2004)] or [3, Laurent & Gregory (2004)]

For a tranche $[A, D]$ with attachment point A and detachment point D in monetary units the normalized cumulative loss of the tranche is defined as

$$\tilde{L}_i^{A,D} = \frac{\min((L_i - A)^+, (D - A))}{D - A}.$$

Let s be the annualized premium, Df_i the (riskless) discount factors and α_i the year fractions. The present values of the premium leg and credit leg are calculated as ¹

$$\begin{aligned}
PV_{PremLeg} &= E\left[s \sum_{i=1}^I Df_i \alpha_i (1 - \tilde{L}_i^{A,D})\right] \\
&= s \sum_{i=1}^I Df_i \alpha_i \left(1 - \int_{\mathbb{R}} E\left[\tilde{L}_i^{A,D} \middle| Z_i^0 = z\right] f_{\mathcal{N}}(z) dz\right), \\
PV_{CreditLeg} &= E\left[\sum_{i=1}^I (Df_i (\tilde{L}_i^{A,D} - \tilde{L}_{i-1}^{A,D}))\right] \\
&= \sum_{i=1}^I Df_i \left(\int_{\mathbb{R}} E\left[\tilde{L}_i^{A,D} \middle| Z_i^0 = z\right] f_{\mathcal{N}}(z) dz \right. \\
&\quad \left. - \int_{\mathbb{R}} E\left[\tilde{L}_{i-1}^{A,D} \middle| Z_i^0 = z\right] f_{\mathcal{N}}(z) dz\right),
\end{aligned}$$

where $f_{\mathcal{N}}$ is density of the standard normal distribution. That means we need to calculate the expected tranche losses at each payment date. The remaining

¹We neglect the accrued for the ease of notation.

computations are trivial.

In the base correlation framework, introduced by [5, Lee McGinty et al (2004)], the tranche losses are calculated by

$$E[\tilde{L}_i^{A,D}] := \frac{D E[\tilde{L}_i^{0,D}] - A E[\tilde{L}_i^{0,A}]}{D - A}$$

where for the equity tranches different correlation are used. Additional constraints maybe added to prevent from arbitrage.

3 Gaussian Copula Model with Stochastic Recovery

To induce stochastic recovery in the Gaussian copula framework we firstly choose for each obligor m a discrete (marginal) recovery distribution R_i^m conditioned on default (i.e. $\tau_m < T_i$)

$$R_{i,\tau_m \leq T_i}^m = \begin{cases} r_1^m & \text{with probability } p_1^m \\ r_2^m & \text{with probability } p_2^m \\ \vdots & \vdots \\ r_J^m & \text{with probability } p_J^m \end{cases} \quad (1)$$

with

$$\begin{aligned} \sum_{j=1}^J p_j^m &= 1, \\ \sum_{j=1}^J p_j^m r_j^m &= \text{REC}^m, \end{aligned}$$

and $r_j^m (j = 1, \dots, J)$ the attainable recovery rates, $p_j^m (j = 1, \dots, J)$ the corresponding probabilities.

While choosing the recovery rate distribution we have to make sure, that the average recovery rate equals the quoted recovery rate REC^m , used for bootstrapping the survival probabilities. It is important to stress the fact, that the probabilities *conditioned on default* have that property, otherwise our model would not be consistent with the single name CDS market and thereby would not price the corresponding portfolio CDS correctly. A feasible distribution is for instance

$$R_{i,\tau_m \leq T_i}^m = \begin{cases} 60\% & \text{with probability } 40\% \\ 40\% & \text{with probability } 30\% \\ 20\% & \text{with probability } 20\% \\ 0\% & \text{with probability } 10\% \end{cases} \quad (2)$$

Of course a finer distribution can be used, which could make sense for bespoke super senior tranches. But for standard tranches this distribution works very well.

An empirical analysis of the recovery rate distribution can be found in [6, Renault, Scaillet (2003)]. They investigate the shape of the recovery rate distributions for different seniorities from historical data and examine their approximations by the Beta distribution. They confirm the intuitive fact that the recovery rate distribution is skewed left for the most senior bonds and skewed right for junior debt. Moreover they conclude that the recovery rate distribution is "far from being Beta distributed". This is fortunately no issue for our model, since the recovery rate distribution can be chosen arbitrarily.

No we induce correlation to the recovery rates due the correlated default triggering factors X_i^m and additional thresholds c_{ij}^m :

$$\begin{aligned} q_{ij}^m &:= q_i^m \left(1 - \sum_{k=1}^j p_k^m\right) \quad \text{for } j = 0, \dots, J \\ c_{ij}^m &:= N^{-1}(q_{ij}^m) \quad \text{for } j = 0, \dots, J \end{aligned}$$

Note that $q_{i0}^m = q_i^m$, $c_{i0}^m = c_i^m$, $q_{iJ}^m = 0$ and $c_{iJ}^m = -\infty$. We define the recovery rate R_i^m for coupon date T_i and asset m via

$$\begin{aligned} R_i^m &:= r_i^m(X_i^m) \\ r_i^m(x) &:= \begin{cases} r_j^m & \text{if } c_{ij}^m < x \leq c_{i,j-1}^m \quad \text{for } j \in \{1, \dots, J\} \\ \text{zero} & \text{else} \end{cases} \end{aligned} \quad (3)$$

The value we set the recovery rate in the "else"-case, actually doesn't matter because it doesn't enter the calculation of the loss distribution. By construction, we ensure that:

$$\begin{aligned} E[r_i^m(X_i^m) | \tau_m < T_i] &= E[r_i^m(X_i^m) | X_i^m < c_i^m] \\ &= \sum_{j=1}^J r_j^m p_j^m \\ &= \text{REC}^m \end{aligned}$$

Hence, the correlation of two recovery rates is induced by the correlation of the default triggering factors $\{X_i^m\}_{m=1, \dots, M}$ in the same way as default correlation is induced. Since the marginal distribution of X_i^m is independent of the factor correlation, the marginal recovery rate distribution R_i^m is also independent of the factor correlation, which is particularly important for a Base correlation framework. An explicit derivation of the recovery correlation can be found in section 4.

The probability of default does obviously not change

$$P_i^m(z) := P(\tau_m \leq T_i | Z_i^0 = z) = \mathcal{N}\left(\frac{c_i^m - \sqrt{\rho}z}{\sqrt{1-\rho}}\right).$$

The probabilities of the recovery rates conditioned on the common market factor Z_i^0 are for $0 < \rho < 1$:

$$\begin{aligned}
p_{ij}^m(z) &\stackrel{\text{def}}{=} P(r_i^m = r_j | Z^0 = z, \tau^m \leq T_i) \\
&= P(X_i^m \leq c_i^m, c_{ij}^m < X_i^m \leq c_{ij-1}^m | Z_i^0 = z) / P_i^m(z) \\
&= P\left(c_{ij}^m < \sqrt{\rho}z + \sqrt{1-\rho}Z_i^m \leq c_{ij-1}^m | Z_0 = z\right) / P_i^m(z) \\
&= P\left(\frac{c_{ij}^m - \sqrt{\rho}z}{\sqrt{1-\rho^2}} < Z_i^m \leq \frac{c_{ij-1}^m - \sqrt{\rho}z}{\sqrt{1-\rho}} | Z_i^0 = z\right) / P_i^m(z) \\
&= \frac{\mathcal{N}\left(\frac{c_{ij}^m - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) - \mathcal{N}\left(\frac{c_{ij-1}^m - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)}{\mathcal{N}\left(\frac{c_i^m - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)}
\end{aligned}$$

For $\rho = 0$:

$$\begin{aligned}
p_{ij}^m(z) &= \frac{\mathcal{N}(c_{ij-1}^m) - \mathcal{N}(c_{ij}^m)}{\mathcal{N}(c_i^m)} \\
&= q_i^m \left(\sum_{k=1}^j p_k^m - \sum_{k=1}^{j-1} p_k^m \right) / q_i^m \\
&= p_j^m
\end{aligned}$$

For $\rho = 1$:

$$p_{ij}^m(z) = \begin{cases} 1 & \text{if } c_{ij}^m < z \leq c_{ij-1}^m \\ 0 & \text{if else} \end{cases}$$

Conditioned on $Z_i^0 = z$ and default we obtain the following recovery rate distribution:

$$R_{i, \tau^m < T_i}^m(z) = \begin{cases} r_1^m & \text{with probability } p_{i1}^m \\ r_2^m & \text{with probability } p_{i2}^m \\ \vdots & \vdots \\ r_J^m & \text{with probability } p_{iJ}^m \end{cases}$$

All is left, is the calculation of the loss distribution. The rest can be handled in the same way as in the standard Gaussian Base correlation model. The Bucketing algorithm in [2, Hull & White (2004)] must be slightly generalised to calculate the loss distribution with stochastic recovery rates. The Bucketing algorithm in [1, Andersen & Sidenius (2004)] can already handle that case.

4 Numerical Results

4.1 Recovery Correlation

In this subsection we derive the default correlation and recovery correlation of two obligors. We present an example for the homogeneous case, where the default probabilities and recovery distributions are equal, and for the inhomogeneous case, where the default probabilities are different. Note that for a portfolio of 125 names there are 7750 of those for each payment date. We first introduce some notations: A pair of default triggering factors X_i^m and X_i^n is bivariate normal distributed with correlation ρ :

$$(X_i^m, X_i^n) \stackrel{d}{=} \mathcal{N}_\rho := \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right).$$

The density is

$$f_{\mathcal{N}}(\rho; x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left(-\frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)} \right).$$

Let

$$\mathcal{N}_\rho(x_0, y_0, x, y) = \int_{x_0}^x \int_{y_0}^y 1 f_{\mathcal{N}}(\rho; x, y) dx dy$$

be the probability that $(X, Y) (\stackrel{d}{=} \mathcal{N}_\rho)$ is inside the specified rectangle. We calculate the recovery correlation conditioned that both names have defaulted. Effectively, that means we have to divide all probabilities by the probability that both names have defaulted:

$$\tilde{\mathcal{N}}_\rho(x_0, y_0, x, y) := \begin{cases} \mathcal{N}_\rho(x_0, y_0, x, y) / \mathcal{N}_\rho(-\infty, -\infty, c_i^m, c_i^n) & \text{if } x < c_i^m, y < c_i^n \\ 0 & \text{if } \textit{else} \end{cases}$$

Let $\tilde{\mathbb{E}} = \mathbb{E}[\cdot | \tau_m \leq T_i, \tau_n \leq T_i]$ be corresponding expectation, where the same notation is used for VAR and COR. Then following holds:

$$\tilde{\mathbb{E}}[R_i^m] = \sum_{j=1}^J r_{ij}^m \tilde{\mathcal{N}}_\rho(c_{ij}^m, -\infty, c_{ij-1}^m, c_i^n) \quad (4)$$

$$\widetilde{\text{VAR}}[R_i^m] = \sum_{j=1}^J (r_{ij}^m)^2 \tilde{\mathcal{N}}_\rho(c_{ij}^m, -\infty, c_{ij-1}^m, c_i^n) - \tilde{\mathbb{E}}[R_i^m]^2 \quad (5)$$

$$\tilde{\mathbb{E}}[R_i^m R_i^n] = \int_{-\infty}^{c_i^m} \int_{-\infty}^{c_i^n} r_i^m(x) r_i^n(y) \tilde{f}_{\mathcal{N}}(\rho, x, y) dx dy \quad (6)$$

$$\begin{aligned} &= \sum_{k=1}^J \sum_{l=1}^J \int_{c_{ij-1}^m}^{c_{ij}^m} \int_{c_{il-1}^n}^{c_{il}^n} r_i^m(x) r_i^n(y) \tilde{f}_{\mathcal{N}}(\rho, x, y) dx dy \\ &= \sum_{k=1}^J \sum_{l=1}^J r_k^m r_l^n \tilde{\mathcal{N}}_\rho(c_{ik-1}^m, c_{il-1}^n, c_{ik}^m, c_{il}^n) \end{aligned} \quad (7)$$

$$\widetilde{\text{COR}}[R_i^m, R_j^m] = \frac{\widetilde{\text{E}}[R_i^m R_j^m] - \widetilde{\text{E}}[R_i^m] \widetilde{\text{E}}[R_j^m]}{\sqrt{\widetilde{\text{VAR}}[R_i^m]} \sqrt{\widetilde{\text{VAR}}[R_j^m]}} \quad (8)$$

From equations (3) and (8) we can conclude directly the not surprisingly fact that the recovery correlation is zero, if the factor correlation is zero. If the factor correlation is one, it does not necessarily mean that recovery correlation is one, which is neither the case for default time correlation.

In Table 1 and Table 2 we calculate the recovery and default correlations and the probabilities that both names default for different factor correlations. We use the recovery rate distribution shown in equation (2). In Table 1 we show the results for different single default probabilities and in Table 2 for equal. The default correlation is defined by $\text{COR}[1_{\{\tau_m \leq T_i\}}, 1_{\{\tau_n \leq T_i\}}]$.

Default Prob1 = 3% , Default Prob2 = 5%			
Factor Correlation	Default Probability	Default Correlation	Recovery Correlation
0%	0.15%	0%	0 %
25%	0.40%	6.76%	3.71 %
50%	0.84%	18.64%	10.24 %
75%	1.58%	38.48%	24.09 %
90%	2.30%	57.97%	45.30 %
95%	2.65%	67.32%	60.24 %
100%	3.00%	76.65%	89.44 %

Table 1: Inhomogeneous case: Correlation in dependency from factor correlation

Default Prob1 = 5% , Default Prob2 = 5%			
Factor Correlation	Default Probability	Default Correlation	Recovery Correlation
0%	0.25%	0%	0 %
25%	0.61%	7.67%	3.93 %
50%	1.22%	20.40%	10.79 %
75%	2.20%	41.07%	25.20 %
90%	3.19%	61.83%	47.19 %
95%	3.71%	72.81%	62.68 %
100%	5.00%	100.00%	100.00 %

Table 2: Homogeneous case: Correlation in dependency from factor correlation

Note that the recovery rate correlation is quite low until 75%. Actually this is a nice result, because we need the impact of recovery correlation primarily for

senior tranches where the copula correlation is usually high. In the homogeneous case we get for a factor correlation of 100% also a recovery and default correlation of 100%, since the default thresholds and factor distributions are identical, and so the recovery distributions.

4.2 Calibration of distressed CDOs

In this subsection we analyse how our new model performs in a distressed CDO market. We chose CDX.IG9 CDO market from 10. March 2008 (see Table 3). At this date, the index was on an all-time high and it was not possible to calibrate the senior tranche 15% - 30% with standard Gaussian Base correlation model.

CDO Market Data CDX.IG9 3/10/2008						
Maturity Reference	5 Years		7 Years		10 Years	
	176		168		163	
Detachment	Spread	Delta	Spread	Delta	Spread	Delta
3%	67.38%	3	70.50%	2.2	73.50 %	1.5
7%	727	3.2	780	3.3	895.5	3.5
10%	403	2	440	2.3	509	2.6
15%	204	1.4	248	1.6	282	1.8
30%	164	1	128.5	1.1	139.5	1.2

Table 3: CDO Market data CDX.IG9 from 10. March 2008

Calibration Results: Base Correlation with Fixed Recovery									
Mat.	5 Years			7 Years			10 Years		
Det	Spread	Delta	Corr	Spread	Delta	Corr	Spread	Delta	Corr
3%	67.38%	2.75	39.99%	70.5%	2.05	40.69%	73.50	1.48	40.47%
7%	727	2.99	65.24%	780	3.11	63.14%	895.5	3.29	57.87%
10%	403	1.88	74.80%	440	2.15	71.87%	509	2.53	65.36%
15%	204	0.81	88.99%	248	1.30	84.69%	282	1.67	78.10%
30%	(201)	1.26	(100%)	(184)	1.11	(100%)	(162)	0.99	(100%)
100%	(56.37)	0.73	–	(59.39)	0.74	–	(63.01)	0.74	–

Table 4: Calibration Results for standard Gaussian Base Correlation

Calibration Results: Base Correlation with Random Recovery									
Mat.	5 Years			7 Years			10 Years		
Det	Spread	Delta	Corr	Spread	Delta	Corr	Spread	Delta	Corr
3%	67.38%	2.97	34.98%	70.5%	2.20	35.73%	73.50	1.59	35.57%
7%	727	3.36	56.84%	780	3.43	55.03%	895.5	3.55	50.17%
10%	403	2.18	64.90%	440	2.41	62.26%	509	2.74	56.14%
15%	204	1.22	77.30%	248	1.59	73.11%	282	1.89	66.60%
30%	115	1.15	90.13%	128.5	0.92	93.08%	139.5	1.12	87.54%
100%	64.38	0.67	–	71.24	0.70	–	67.96	0.65	–

Table 5: Calibration Results for Base Correlation with Stochastic Recovery

Table 4 contains the calibration results for the standard Gaussian Base correlation model. Model quotes and corresponding Base correlation which does not match the market quotes are in brackets. The model fails to calibrate the 15% - 30% tranche for all maturities, thereby also the deltas look strange in 5y.

In Table 5 we present the calibration results of our new model with stochastic recovery. We used again the recovery rate distribution from equation (2). We can observe following improvements over the standard model. Firstly, the calibration works for all tranches. Secondly, the deltas are closer to the market data. Thirdly, The Base correlation levels are lower than in the standard case, which widens the calibration range for the senior tranches. Finally, the deltas are higher than in the standard Gaussian Base correlation, which is a common observation from more sophisticated valuation models.

5 Conclusion

We present a Gaussian Copula CDO model with stochastic recovery rates. By including additional "recovery thresholds" we induce recovery correlation driven by the usual factor correlation. Due to this extension, our model can be calibrated to the complete set of CDO tranche quotes also in distressed scenarios. Moreover, in our calibration example the deltas were closer to the market deltas as in the standard model. Since the recoveries are stochastic, portfolio losses of up to 100% are possible (while preserving the mean recovery), and so this model can also be used to handle super senior tranches like 60% to 100%. To realize this, just a Base correlation at 60% must be included in the calibration. The model is a practical extension of the well established Base correlation model, so it has the same drawbacks (which are mainly of theoretical nature) and all its benefits (which are mainly of practical nature). The calculation time of our model is not more than doubled in comparison to the standard model. The extension is obviously not restricted to Gaussian copulas and can also be used within other copulas models like Levy as well as Random Factor Loading models.

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